Indian Statistical Institute
Second Semester Examination 2003-2004
M.Math I Year
Differential Geometry

Time: 3 hrs Date:03-05-04 Max. Marks : 50

Answer all six questions

Possibly Useful Formula:

$$2\langle \nabla_X Y, Z\rangle = X\langle Y, Z\rangle + Y\langle X, Z\rangle - Z\langle X, Y\rangle + \langle [X,Y], Z\rangle - \langle [X,Z], Y\rangle - \langle [Y,Z], X\rangle$$

1. Consider the two-dimensional Riemannian manifold (M, g) where $M = \{(x,y) \in I\!R^2 : y > 0\}$ and $g = \frac{1}{y^2} dx \otimes dx + \frac{1}{y^2} dy \otimes dy$

   a) For any $a \in I\!R$, prove that the map $f_a : M \to M$ given by $f_a(x,y) = (x+a,\ y)$ is an isometry.

   b) For any $a \in I\!R$, prove that the curve $\sigma_a : I\!R \to M$ given by $\sigma(t) = (a,\ e^t)$ is a geodesic.

   c) Calculate the sectional curvature of $(M,\ g)$ at any point $(x_0,\ y_0)$. [11]

2. Let M and N be n-manifolds with M compact and N connected. Let $f : M \to N$ be an immersion. Prove that f is onto (surjective). [6]

3. Let $\pi : (\tilde{M},\ \tilde{S}) \to (M,\ g)$ be a Riemannian covering i.e., a smooth covering such that π is a local isometry. Assume that $(M,\ g)$ and $(\tilde{M},\ \tilde{g})$ are oriented. If the covering is a finite k-sheeted covering prove that

$$\mathrm{Vol}(\tilde{M}, \tilde{g}) = k\ \mathrm{Vol}(M, g),$$

   where "Vol" denotes volume. [8]

   Hint: Recall the proof when π is actually an isometry.

4. Let M be a compact oriented n- manifold and N any n-manifold. Let Ω be any n-form on N and let $f,\ g : M \to N$ be two smooth maps. If f and g are smoothly, homotopic, (i.e. if there is a smooth map $F : M \times [0,1] \to N$ with $F(x,0) = f(x)$ and $F(x,1) = g(x)$ for all x). Then prove that $\int_M f^*(\Omega) = \int_M g^*(\Omega)$ [8]

5. Let (M,g) be a Riemannian manifold and let A be the image of a closed geodesic in M. Let p be a point in M which is not in A. Let $C : [0, L] \to M$ be a geodesic (parametrised by arc-lenth) such that $l(c) = \inf_{x \in A} d(p,x)$

a) Let C_t be a variation of C and Y be the corresponding variation vector field. We know that $\inf_{-\epsilon<t<\epsilon} l(C_t) = l(C)$ for all variations with $C_t(L) \in A$ and $C_t(0) = p$. For variations of this type, what are the restrictions on $Y(0)$ and $Y(L)$?

b) The first variation formula for the length functional is

$$\frac{d}{dt} l(C_t)\Big|_{t=0} = \langle Y(s),\ C'(s)\rangle\Big|_0^L - \int_0^L \left\langle Y(s), \nabla_{C'(s)} C'(s)\right\rangle ds$$

Using the above formula, prove that $\langle C'(L), X\rangle = 0$ for any $X \in T_{c(L)}A$. Clearly state any result you use. [6]

6. Let $f : (M, g) \to \mathbb{R}$ be a smooth function on a Riemannian manifold. The Hessian of f at a point $p \in M$ is a 2-tensor denoted by $D^2 f_p$ and defined as follows: Let $X, Y \in T_pM$ and let $\tilde{X}, \tilde{Y}$ be vector fields extending X, Y. Then

$$D^2 f_p(X, Y) := \tilde{X}_p(\tilde{Y}(f)) - (\nabla_{\tilde{X}} \tilde{Y})_p f$$

a) Prove that $D^2 f$ is actually a tensor, i.e., $D^2 f_p(X, Y)$ doesn't depend on the extensions $\tilde{X}$ and $\tilde{Y}$.
b) Prove that $D^2 f$ is a symmetric tensor.
c) If p is a local minimum of f, prove that $D^2 f_p(X, X) \geq 0 \ \ \forall X \in T_pM$.
[11]

Hint: Consider a geodesic σ with $\sigma(0) = p, \ \ \sigma'(0) = X$.